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Matrix Decompositions In Graph Theory: A Comprehensive Study

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ABSTRACT

Matrix decompositions are powerful tools in graph theory, providing critical insights into the structure and properties of graphs. Key decompositions such as the adjacency matrix, Laplacian matrix, and their variants facilitate various graph-related computations and analyses. Advanced decompositions, like those involving the graph's incidence matrices and their factorizations, further enhance our understanding of graph properties and optimization problems. These techniques underpin algorithms for graph partitioning, network analysis, and dynamic graph models, making matrix decompositions indispensable for both theoretical and applied graph theory. We advocate a useful vision of matrix decomposition troubles on graphs consisting of geometric matrix accomplishment and graph regularized dimensionality discount. Our consolidate framework is primarily based on a key idea that using reduced basis to symbolize a feature on the result space of graph is sufficient to get better a short rank matrix estimation even from a sparse signal.

INTRODUCTION

Matrix decompositions are a powerful tool in graph idea, offering perception into diverse structural houses of graphs and allowing efficient algorithms for problems associated with graphs. Here's a top level view of key matrix decompositions and their packages in graph theory.

The assumption that high-dimensional information samples lie on or near an easy low-dimensional manifold is exploited as a regularizer or prior in many gadget mastering algorithms. Often, the low-dimensional manifold statistics is exploited through a graph shape among the records samples. As an end result, graph is regularly used as a regularizer in various gadget gaining knowledge of problems consisting of Dimensionality reduction, Hashing or Matrix of entirety to call some. In this article, we cognizance on Dimensionality discount and matrix completion and advise a principled framework that offers a unified option to each these troubles with the aid of modelling the more geometric facts to be had in phrases of graphs.

Dimensionality reduction: Given a data matrix $M \in \mathbb{R}^{m \vee n}$ with n m-dimensional data vectors, most prior work related to PCA can be broadly categorised in two themes:

1) matrix factorization approach of the classical PCA and its variants

2) matrix subtraction approach of robust PCA and its variants. The former learns the projection $Q \in \mathbb{R}^{d.n}$ of M in a d-dimensional linear space characterized by an orthonormal basis $U \in \mathbb{R}^{m.d}$ Several follow up works have shown that the clustering quality of PCA can be significantly improved by incorporating the data manifold information in the form of some underlying graph structure.

Instead of relying on matrix factorization, the second one line of work directly estimates easy low rank approximation X of facts matrix M by way of keeping apart noise with a matrix additive version. Along those line, Fast Robust PCA on graphs introduces a joint notion of low rankness for the rows and columns of a records matrix and proposes to mutually limit the Dirichlet power at the row and column graphs:

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$$\min_{X} \| M - X \|_1 +_{\gamma 1} tr(XL_1 X^1) +_{\gamma 2} tr(X \quad L_2 X)$$

Here L_1, L_2 are Laplacian matrix of graphs constructed, respectively, from the rows and columns of the statistics matrix M. Conceptually, minimizing the Dirichlet strength $tr(XL_1X^{T)}$ advance smoothness of X through penalizing big frequency components of the signals on corresponding graphs. The writer of FRPCAG demonstrate theoretically that underneath positive assumptions this minimization is hooked up with the spectrum of the basic short rank matrix X. Building in this idea, we as an alternative without delay constrain the short rank estimation with the aid of decomposing it the use of the first few eigenvectors of row and column graph Laplacians $X = \Phi C \Psi^T$ and optimizing over the coupling matrix C at most. Our approach, comparable in spirit to the matrix factorization method, ends in explicit manage over the ensuing rank of the matrix, and thereby, superior overall performance and drastically less complicated optimization troubles.

Matrix completion offers with the recuperation of lacking values of a matrix of which we have best measured a subset of the entries. In widespread, with none constraints, this hassle is sick-posed. However if the rank of underlying matrix is small, the number of tiers of freedom decreases and for that reason, it is common to locate the bottom rank matrix that has the same opinion with known measurements. Under this low rank assumption, the trouble is very similar to dimensionality discount and can be rewritten as,

$$\min_{X} \quad rank(X) \mid \frac{\mu}{2} \|(X - M) \Theta S\|_F^2$$

Here $\boldsymbol{X}$ stands for the unknown matrix, $M \in \mathbb{R}^{m-n}$ for

The ground truth matrix, S is a binary masks representing the enter assist, and $\odot$ denotes the Hadamard product. Various issues in collaborative filtering may be posed as a matrix of entirety trouble, wherein as an instance the columns and rows constitute customers and items, respectively, and matrix values represent a rating figuring out the desire of user for that item. Often, extra structural information is to be had in the form of column and row graphs representing similarity of customers and objects, respectively. Most of the earlier paintings that carries geometric shape into matrix of completion issues is either based totally on quite engineered frameworks, e.g., or non-convex components with several hyperparameters thereby making the overall optimization more difficult to optimize. Instead, our simple system based at the functional map representation, inclusive of a unique regularizer, mitigates the problems related to.

Contributions. Our contributions are threefold. First, we suggest a singular unified vision of geometric matrix crowning glory and graph regularized dimensionality trimming this is convex and clean. Second, intellectually, our matrix decomposition components absolutely imposes and optimizes for an extremely low rank approximation and, as we exhibit beneath, is empirically extra correct in recovering a short rank matrix estimation than competitive baselines. Third, we advise a unique regularization influenced from the functional map writings that is shown to be aggressive with a combination of numerous regularizers on numerous real world datasets.

ADJACENCY MATRIX DECOMPOSITION

Graphs appearing in this paper are always assumed to be finite, undirected and simple.

1.1. Adjacency matrices. Given a graph X on the vertex set V*(G)* with the edge set *E(G),* the adjacency matrix *A(X)* of X is a | V(G)| x | V(G)| matrix defined by

$$A(X)_{uv} = \begin{cases} 1 \ if \ uv \ \in E(G), \\ 0 \ if \ uv \ \notin E(G). \end{cases}$$

Adjacency matrices of graphs and their spectrum often give a characterization of several properties on graphs. Some variation on the universal values of the eigenvalues of the adjacency matrices and their determinants are shown. Moreover, specified a graph X, let $S(X)$ = j - I - 2A(X), in which I is the identification matrix and I is the all-one matrix. The symmetric matrix S(X) is called the Seidel matrix of a graph X. This has a few connections with different combinatorial gadgets. For instance, the eigenvalues of Seidel matrices are used for the investigations or the characterizations of equiangular traces and strongly everyday graphs. For extra data, please consult, e.G..

On those research, adjacency matrices and Seidel matrices are treated as the matrices over $\mathbb{R}$. On the alternative hand, there are some studies at the adjacency matrices over a finite area as far because the authors recognize. The aim of this research is to produce the studies on adjacency matrices over a finite prime subject.

1.2. Prime fields and quadratic residues. For a prime p, let $\mathbb{F}_p$denote the prime field of order p and let $\mathbb{F}^X_{p=} \mathbb{F}_p \backslash \{0\}.$ For $\mathbb{F}_p$, let S(p) (resp. $\mathcal{T}(p)$) be the set of quadratic residues (resp. quadratic nonresidues) of $\mathbb{F}_p$. Namely,

$$S(p) = \{a^2 : a \in \mathbb{F}^X_p\} \ and \ \tau(p) = \ \mathbb{F}^X_p \ \ S(p)$$

In particular, $\mathbb{F}^X_p = S(p) \sqcup \ \tau(p)$. For example, we listthe quadratic

(non)residues of $\mathbb{F}_p$for some small odd primes:

p	$\mathcal{S}(p)$	$\mathcal{T}(p)$
3	1	−1
5	±1	±2
7	1, 2, −3	−1, −2, 3
11	1, −2, 3, 4, 5	−1, 2, −3, −4, −5
13	±1, ±3, ±4	±2, ±5, ±6

TABLE 1. Examples of quadratic (non)residues

Regarding quadratic residues of $\mathbb{F}_p$, the following facts are well known:

- If $p \geq 3, then \ |S(p)| = \ |\tau(p)| = (p-1)/2.$
- For $x, y \in \mathbb{F}^X_p, if \ x, y \in S(p) or \ x, y \in \tau(p), then \ xy \in S(p).$
- $If \ x \in S(p) \ and \ y \in \tau(p), then \ xy \in \tau(p).$
- $x \in S(p) \ if \ and \ only \ if \ x^{-1} \in S(p).$

For more details, please consult, e.g.,.

1.3. Direct sum decompositions of symmetric matrices. Given two square matrices M and M', let $M \oplus M'$ denote the direct sum of M and M', i.e., $\oplus \ M' = \begin{pmatrix} M & \\ & M' \end{pmatrix}$. Moreover, we use the notation

$$nM := \underbrace{M\oplus \ldots \oplus M.}_{n}$$

Let k be a field. For symmetric matrices M and M' whose entries belong to k, we say that M and M' are *similar* over k if there is a regular matrix P over k such that $M'=P^tMP$, where P^t denotes the transpose of P. We use the notation like $M \sim M'$ if M and M'

are similar, or $M \underset{p}{\to} M'$ if $M' = P^t$ MP. It is well familiar that any symmetric matrix is near over R to a definite diagonal matrix.

In what follows, we abuse the comment for the adjacency matrix of X as the same symbol of a graph X.

Specified a graph X, we say that X *can be decomposed into* $X_1, \ldots, X_s$ over a field k, where $X_1, \ldots, X_s$ are graphs, if X is similar over k to a direct sum of $X_1's, i.e., X \sim n_1X_1 \oplus \ldots \oplus n_sX_s$ for some $i \in \mathbb{Z}_{\geq 0}$.

The rank of the adjacency matrix over $\mathbb{F}_2$ is studied. By using the discussions there, we can claim the following:

Theorem 1. Any graph X can be decomposed into K_i and K_2 over $\mathbb{F}_2$.

Here, K_n denotes the complete graph on *n* vertices The goal of this paper is to develop the similar result to Theorem 1 in the case of odd primes.

1.4. Main Results. We divide the statements of our main theorems into six cases as shown in the table below.

	$-1, 2 \in \mathcal{S}(p)$	$-1 \in \mathcal{S}(p), 2 \in \mathcal{T}(p)$	$-1 \in \mathcal{T}(p)$
$3 \in \mathcal{S}(p)$	Theorem 4	Theorem 3	Theorem 2 (3)
$3 \in \mathcal{T}(p)$	Theorem 2 (1)	Theorem 2 (2)	Theorem 2 (4)

TABLE 2. Division by six cases of main theorems For example, the primes satisfying each condition are as follows:

	$-1, 2 \in \mathcal{S}(p)$	$-1 \in \mathcal{S}(p), 2 \in \mathcal{T}(p)$	$-1 \in \mathcal{T}(p)$
$3 \in \mathcal{S}(p)$	73	13	11
$3 \in \mathcal{T}(p)$	7,17	5	19

TABLE 3**.** Examples of primes of Table 2

Theorem 2. Let p be an odd prime. Then the following assertions hold:

1.4. If $-1,2 \in S(p)$ and $3 \in \tau(p)$, then any graph can be decomposed into Ki, K_2, K_3 and K_4 over $\mathbb{F}_p$.
1.5. If $-1 \in S(p)$ and $2,3 \in \tau(p)$ then any graph can be decomposed into K_i, K_2, K_3, K_4 and B over $\mathbb{F}_p$.
1.6. If $p \geq 5, 3 \in S(p)$ and $-1 \in \tau(p)$, then any graph can be decom posed into K_i, K_2, K_3, K_4 and C_5 over $\mathbb{F}_p$.
1.7. If $p \geq 5$ with $-1,3 \in \tau(p)$ or p = 3, then then any graph can be decomposed into K_i, K_2, K_3 and C_5 over $\mathbb{F}_p$.

Here, C_5 denotes the cycle of length 5 and B denotes the graph with 5 vertices depicted in Figure 1.

Theorem 3. Let $p \geq 5$ and assume that $-1,3 \in s(p)$ and $2 \in \tau(p)$.

Then any graph can be decomposed into K_i, K_2, K_3, B and X_6 over $\mathbb{F}_p$, where

$X_6 := K_6\ if\ 5 \in \tau(p), X_6 := D\ if\ 7 \in \tau(p)$, and X_6 is not required otherwise.

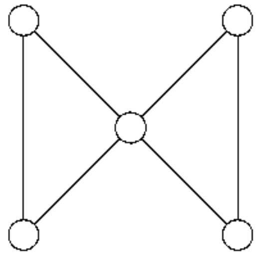

Figure 1 : Graph B

Here, D is the following graph with 6 vertices (Figure 2).

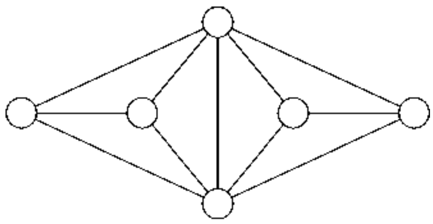

Figure 2: Graph D

Theorem 4. Let $p \geq 5$and assume that $-1,2,3 \in S(p)$ Then there exist

graphs X_4 and X_5 such that any graph can be decomposed into K_i, K_2, K_3, X_4 and X_5 over $\mathbb{F}_p$.

GRAPH CUTS AND DECOMPOSITIONS

Graph cuts have emerged as an increasingly powerful tool for exact or approximate energy minimization for various discrete pixel labeling problems in computer vision such as image segmentation, video segmentation, image restoration, stereo, shape reconstruction, object recognition, texture synthesis, and others. One of the main reasons at the back of their developing recognition is the availability of green algorithm with low computational complexity to clear up the min-cut/max-glide problem in graphs, which in turn lets in for the computation of worldwide most advantageous solutions for certain important energy functions in vision. In Greig et al. (2019) carried out graph cuts to optimize this kind of power characteristic and given a globally ultimate binary labeling in laptop imaginative and prescient. Previously, specific minimization of energy features in Greig et al. (2019) was not possible and such energies had been approached with the aid of particularly using iterative algorithms like simulated annealing, whose effects generally have been a long way from the global minimal. Boykov et al. (2017) then provided two algorithms based on graph cuts that successfully find a local minimum with respect to 2 forms of big actions, namely enlargement moves and swap actions for the optimization hassle of multi-labeling. The graphs similar to those applications are commonly huge 2D or 3D grids and the efficiency of the min-cut/max-glide algorithms is an problem that need to be considered. Recently, graph-cut algorithms have been extended to solve some minimization problems from PDE-based image processing and these problems have been traditionally solved through Euler-Lagrangian minimization approaches. Graph-cut method was used to get fast solutions for some TV (total variation) minimization problems. In some new attempts, continuous max-flow approaches have been proposed. These approaches is extending the max-flow problem from discrete settings to continuous settings and some primal-dual algorithms can be used to get fast numerical schemes. The goal of this paper is to develop a recursive graph cuts method which can solve the min-cut/max-flow problem more efficiently than the existing methods. The proposed method is essentially a generic framework into which any existing mincut/max-flow algorithm can be integrated to produce a recursive method with lower time complexity than the original one. Especially, the solution by the recursive framework is exactly the same with the original minimum cuts algorithm, not an approximation. If the time complexity of the integrated min-cut algorithms can be denoted as $O(N^p)$, we can prove the total time complexity of the proposed RMC algorithm is $O\left(N^{\frac{\log 3}{\log 2}}\right)$. This is implemented based on the proposed Graph decomposition technique and minimum cuts composition theorem. The primary conclusions of the proposed method are as follows.

1) Given an arbitrary cut C_0 of G, we build subgraphs G_1 and G_2. assume that the min-cuts of G_1, G_2 and G are $c_1(S_1, T_1)$, $c_2(S_2, T_2)$ and c(S, T) respectively. We prove that

$$S_1 S \text{ and } T_2 \subseteq T$$

This justifies that the min-cut Cmin of G lies in the gray zone.

2) We further construct subgraph G_3 according to the min-cuts $c_1(S_1, T_1)$ and $c_2(S_2, T_2)$ of subgraphs G_1 and G_2. Assume that the min-cut of G_3 is $c_3(S_3, T_3)$. We prove that

$$S_3 \subseteq S \text{ and } T_3 \subseteq T$$

This means that the min-cut of G_3 is a subset of min-cut of G. This argument directly leads to the conclusion that

$$S = S_1 + S_3 - s \text{ and } T = T_2 + T_3 - t$$

Therefore, we can obtain c(S, T) by computing $c_1(S_1, T_1)$, $c_2(S_2, T_2)$ and $c_3(S_3, T_3)$.

3) Our proof and the recursive cut method are independent of the minimum cut algorithms. We don't need to convert between the excess or the deficit and tlinks which is just suitable to Push- Relabel algorithms. We can apply any min-cut/max-flow algorithm, including "augmenting paths" style algorithms and "push-relabel" style algorithms to our general graph decomposition and minimum cuts composition framework.

4) Since our proposed method is independent of the min-cut/max-flow algorithm, we can further apply this method to the sub-graphs G_1, G_2 and G_3 of G. Therefore, a recursive frame can be draw to resolve the min-cut/max-flow problems. Theoretically, the recursive procedure can be pursue until the constructed sub-graphs include only one node in addition to the source and sink nodes. We develop a new recursive minimum cut (RMC) framework to resolve max-flow/min-cut problem. We show that the mean time complexity is $O\left(N^{\frac{\log 3}{\log 2}}\right)$

5) Based on the graph decomposition method and minimum cuts composition theorem, a parallel algorithm can be developed to solve the min-cut problem with time complexity of order O(N) using N processors.

GRAPH DECOMPOSITION AND FACTORIZATION

Graph decomposition and factorization are techniques used to simplify, analyze, and solve problems related to graphs by breaking Them down into greater doable or significant com ponents. In the recent years, matrix factorization approaches have grown to be an critical device in gadget studying and located a success application for obligations like, e.g., facts retrieval, data mining and pattern reputation. Well-recognized strategies primarily based on matrix factorization are Nonnegative Matrix Factorization (NMF), Principal Component Analysis (PCA), Singular Value Decomposition (SVD) and Latent Dirichlet Allocation. Relations among matrix factorization and clustering had been hooked up, shedding light on a selection of different programs. Matrices present process a factorization in studying algorithms generally rise up as ordered collections of function factors, or as representations of the similarity/dissimilarity family participants among facts gadgets. In the latter case, we are capable of without problems interpret the matrix as the adjacency matrix of a weighted graph having the records items as vertices. The software program of matrix factorization for the evaluation of graphs is specially restrained to clustering. This paper pursuits at giving a singular viewpoint approximately the position of Matrix factorization in the evaluation of graphs and, within the particular, we show how matrix factorization can serve the motive of compressing a graph. Compressing records is composed in changing its representation in a manner to require fewer bits.

Depending at the reversibility of this encoding procedure we'd have a lossy or lossless compression. Information-theoretic works on compressing graphical structures have these days seemed. However, they do now not consciousness on keeping a graph shape as the compressed illustration, which is alternatively what we intention at in our graph compression version. Our work is as an alternative nearer in spirit, that is although confined to unweighted graphs,. Moreover, it is associated with the Szemeredi regularity lemma.

A famous result in extremal graph principle, which roughly states that a dense graph may be approximated into a bounded number of random bipartite graphs. An algorithmic version of this lemma has been used for speeding-up a pairwise clustering algorithm. A hassle related to graph compression that has targeted the eye of researchers within the network and sociometric literature for the final a long time is blockmodeling. Blockmodels try to institution the graph vertices into corporations that maintain a structural equivalence, i.e. Vertices falling inside the equal institution should show off similar family members to the nodes in other corporations (inclusive of self-similarity), and they range by way of the way in which structural equivalence is defined. We refer to for an overview of block models for recent developments on mixed-membership stochastic block models. The solution that we propose to compress a graph can be regarded as a block model where blocks and their relationships can be determined using a matrix factorization approach.

CONCLUSION

Matrix decompositions play a crucial role in graph theory by providing deep insights into graph properties and enabling efficient algorithmic solutions. Whether through eigenvalue decompositions, factorization techniques, or combinatorial matrix decompositions, these methods help in analyzing and solving various graph-related problems, from connectivity and clustering to network flow and optimization.

In this work, we provide A unique unified view of geometric matrix finishing touch and graph regularized dimensionality reduction and set up empirically and theoretically that the usage of a reduced foundation to represent a characteristic at the product area of graphs already gives a sturdy regularization, this is sufficient to get better a low rank matrix approximation. Moreover, we advocate a novel regularization and show, through sizeable experimentation on real and synthetic datasets, that our unmarried regularization may be very aggressive when compared to the combination of several different regularizations proposed earlier than for geometric matrix finishing touch hassle. Extracting geometric facts from graph based facts is a middle undertaking in numerous domains from few shot gaining knowledge of, 0 shot studying in computer vision, gadget mastering to understanding graph based totally problems in natural language processing considering that graphs seem everywhere. For future work, we plan to extend our framework to numerous such huge scale issues and additionally take a look at its robustness to noise and corruptions in input data.

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